# BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY.

CONTINUATION OF THE BULLETIN OF THE NEW YORK MATHEMATICAL SOCIETY.

A HISTORICAL AND CRITICAL REVIEW OF MATHEMATICAL SCIENCE.

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VOL. IV.

OCTOBER 1897 TO JULY 1898.

PUBLISHED FOR THE SOCIETY
BY
THE MACMILLAN COMPANY,
LANCASTER, PA., AND NEW YORK,
1898.

Reprinted 1968

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BULLETIN OF THE

NEW YORK MATHEMATICAL SOCIETY

## THE FOURTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE Fourth Summer Meeting of the AMERICAN MATHEMATICAL SOCIETY was held at the University of Toronto, Toronto, Canada, on Monday and Tuesday, August 16–17, 1897, thus immediately following the adjournment of the American Association for the Advancement of Science from Detroit to Toronto, and immediately preceding the Toronto Meeting of the British Association. Under these favorable conditions of scientific environment the success of the meeting was assured at the start. The actual outcome, however, exceeded all anticipation. In number and value of papers presented, in attendance of members and distinguished guests, and in scientific interest and enthusiasm, this meeting will rank among the most important scientific gatherings that have taken place on this continent, and is a distinct landmark of the vigorous growth of the Society and the scientific activity of its individual members.

Fifty-five persons were registered as in attendance, of whom forty-four were members of the Society. Twenty-one papers were read, all, with one exception, being by members of the Society. The list of those present is as follows: —

Professor Alfred Baker, Dr. Charlotte C. Barnum, Professor W. W. Beman, Dr. E. M. Blake, Professor F. H. Bigelow, Professor Oskar Bolza, Professor F. N. Cole, Professor A. T. DeLury, Dr. L. W. Dowling, Professor W. P. Durfee, Professor H. T. Eddy, Professor T. W. Edmondson, Professor Thomas S. Fiske, Professor A. R. Forsyth, Mr. J. C. Glashan, Dr. J. W. Glover, Professor A. G. Greenhill, Dr. Harris Hancock, Professor A. S. Hathaway, Professor Ellen Hayes, Professor O. Henrici, Dr. G. W. Hill, Professor T. F. Holgate, Dr. J. I. Hutchinson, Mr. H. G. Keppel, Professor J. E. Kershner, Professor P. A.

Lambert, Dr. G. H. Ling, Professor H. Maschke, Mr. W. McCabe, Dr. Emory McClintock, Mr. A. C. McKay, Mr. J. C. McLellan, Professor J. McMahon, Professor A. Macfarlane, Professor W. H. Metzler, Professor H. B. Newson, Professor Simon Newcomb, Mr. R. L. Sackett, Professor A. W. Scott, Mr. F. Shack, Professor J. B. Shaw, Mr. F. J. Smale, Dr. V. Snyder, Professor J. H. Tanner, Professor H. W. Tyler, Professor E. B. VanVleck, Professor C. A. Waldo, Professor A. G. Webster, Professor P. Wernicke, Mr. A. E. Western, Professor H. S. White, Mr. E. T. Whittaker. Professor R. S. Woodward, and Professor Alexander Ziwet.

Two sessions were held each day, at 10 A. M. and 2:30 P. M. The President, Professor Simon Newcomb, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. Herbert G. Keppel, Northwestern University, Evanston, Ill.; Mr. Cassius J. Keyser, Columbia University, New York, N. Y.; Professor William H. Maltbie, The Woman's College, Baltimore, Md. Three applications for membership were received. It was announced that the next Summer Meeting of the Society would be held at Boston, Mass., at or about the time of the meeting of the American Association for the Advancement of Science.

At the conclusion of the meeting resolutions were adopted tendering the thanks of the Society to the University of Toronto and its officers for many courtesies shown to the Society.

The following papers were read at the meeting:

(1) Dr. E. M. BLAKE: "Upon the representation by ruled surfaces of the curves drawn by mechanisms. Preliminary report, illustrated by models."

(2) Dr. L. W. DOWLING: "A contribution to the theory of plane curves."

(3) Professor ELLEN HAYES: "Note on the Folium of Descartes."

(4) Professor THOMAS F. HOLGATE: "A geometrical locus connected with a system of coaxal circles."

(5) Professor P. WERNICKE: "On the solution of the map-color problem."

(6) Professor H. B. NEWSON: "On the Riemann-Helmholtz-Lie problem of the foundations of geometry."

(7) Professor J. B. SHAW: "Quaternion invariantive operators."

(8) Dr. VIRGIL SNYDER: "The geometry of some differential expressions in hexaspherical coördinates."

(9) Professor E. B. VAN VLECK: "On certain differential equations of the second order allied to Hermite's equation."

(10) Professor OSKAR BOLZA: "Concerning the cubic involution and the cubic transformation of elliptic functions."

(11) Dr. J. C. FIELDS: "The determination of the rational function in the reduction of the general Abelian integral to the sum of a rational function and a fundamental system of elementary integrals."

(12) Dr. J. I. HUTCHINSON: "On the reduction of hyperelliptic functions ($p = 2$) to elliptic functions by a transformation of the second degree."

(13) Dr. EMORY MCCLINTOCK: "Further researches in the theory of quintic equations."

(14) Professor H. MASCHKE: "A theorem concerning the coefficients of linear substitution groups of finite order with $n$ variables."

(15) Dr. G. A. MILLER: "On the commutator groups."

(16) Professor H. S. WHITE: "Collineations in a plane with invariant conic or cubic curves."

(17) Professor E. H. MOORE: "Concerning triple systems."

(18) Mr. F. W. FRANKLAND: "Theory of discrete manifolds."

(19) Mr. P. H. PHILBRICK: "The true transition curve."

(20) Dr. ARTEMAS MARTIN: "About sixth power numbers whose sum is a sixth power."

(21) Professor A. S. HATHAWAY: "Preliminary report on alternate functions of complex numbers."

The mechanisms considered in Dr. Blake's paper consist of a fixed plane $m_0$ with three coincident moving planes, $m_1$, $m_2$, $m_3$. To both $m_0$ and $m_2$ are connected both $m_1$ and $m_2$. The connections are of two kinds; first two planes may have a point in common about which they are free to rotate, or, second, they may have a straight line in common along which they are free to slide. If connections of the first kind only are used we have the well-known three-bar mechanism. Using both kinds of connections, there are ten possible types of mechanisms, but of which four are trivial. If to $m_2$ there be rigidly attached a straight line $l$, making with it an acute angle, then by the motion of $m_2$ due to the mechanism, $l$ will describe a ruled surface, such that any section parallel to $m_0$ will be the curve drawn by a point of $m_2$ in the projection of $l$ upon it.* A number of string models illustrating these surfaces were shown.

* A device similar to this was used by S. Roberts (*Proc. London Math. Soc.*, vol. 3, 1871) in a general investigation "On the motion of a plane under certain conditions."

In his paper on a contribution to the theory of plane curves, Dr. Dowling starts out from Zeuthen's article "*Sur les differentes formes des courbes planes du quatrième ordre*" (*Math. Ann.*, vol. 7, p. 410; 1874), where it was shown that a quartic curve necessarily possesses four real bitangents of the first kind. This result was extended by Professor Klein (Cf. *Math. Ann.*, vol. 10, p. 199) to curves of order $n$; viz., the following formula is true,

$$w' + 2t'' = n(n-2),$$

where $w'$ is the number of real inflexions and $t''$ the number of isolated bitangents which the curve possesses. The object of this paper is to show, in the first place, that a curve of order $n$ must necessarily possess $\frac{n(n-2)}{2}$, if $n$ is even, or $\frac{n(n-2)-3}{2}$, if $n$ odd, real bitangents of the first kind, that it may have more but cannot have fewer than this number. This is shown by considering the possible deformations which the curves of a sheaf of order $n$ may undergo. In the second place, the equation of any curve of order $n$ may be put in the form,

$$\alpha\beta\, U^{(0)} - \varphi^2 \prod = 0$$

in a finite number of ways, where $\alpha = 0$ and $\beta = 0$ are the equations of two real bitangents, $U^{(0)} = 0$ is the equation of a curve of order $n-2$ possessing a node, $\varphi = 0$ is the equation of a conic, and $\prod = 0$ is the combined equations of $n-4$ straight lines. In the third place it follows from the above equation that a curve of order $n$ may be regarded as the envelope of a special linear system of curves of order $n-2$. The theory of poles and polars may be extended to the general curve of order $n$ as has been done for the quartic. Cf. Salmon: *Higher Plane Curves*, p. 224.

Professor Hayes' paper on the Folium of Descartes contains a comparative study of a curve allied to the Folium, and an examination of a surface suggested by the two curves.

Professor Holgate discusses a system of coaxial circles in connection with a sheaf or pencil of rays. Let there be given in a plane a sheaf or pencil of rays through a point $P$ and a coaxial system of circles having real intersections $A$ and $A'$. Each ray of the sheaf is tangent to two circles of the coxial system and to each circle of the system two rays of the sheaf are tangent. Choosing then any circle of

the system and one of its tangent rays, the second circle of the system to which this ray is tangent is immediately determined, then the second ray of the sheaf tangent to this second circle, and again a third circle tangent to this second ray and so on continuously. Restricting the first two circles to equality, and consequently the first ray to be parallel with the line of centres, it is found that the circuit of circles and rays will close with three circles and three rays if only the point $P$ lies upon one of the parabolas having the line of centres for directrix and the point $A$ or $A'$ for focus. The centre of the third circle is determined by the straight line joining $P$ and $A$ or $A'$.

The general problem of Professor Wernicke's paper is immediately reducible to the following: To color, with four tints, a map in each vertex of which only three districts concur. If a map has been colored with four tints its frontiers may be marked with three indices so that concurrent frontiers have different indices. The converse of this is to be proven for surfaces of deficiency 0. Given a map correctly colored and with its frontiers marked, the author proves that any triangles, quadrangles, and pentagons can be introduced and correctly marked at the same time. The main theorem then follows by induction. Several corollaries are obtained, among them that any polyhedron in each summit of which three edges meet can by mere (triangular) truncation of single summits be changed into a polyhedron, every face of which has for the number of its sides a multiple of three.

Professor Newson discusses the Riemann-Helmholtz problem as stated by Lie, viz.: To find the properties which are common to the group of Euclidian and the two groups of non-Euclidian motion, and by which these three groups are distinguished from all other possible groups of motion. This paper treats only of the first part of the problem. The three groups of motions are defined with reference to the Cayleyan Absolute and all the real subgroups determined for each case. These subgroups are then classified with reference to the thirteen types of projective transformations in space. The types of motion corresponding to these types of projective transformations are illustrated by figures. The properties of the different types of motions are stated for the three kinds of space, and the corresponding types for the three spaces are compared. The properties common to the group of Euclidian and the two groups of non-Euclidian motions are thus found. The following results among others are reached: The most general type

of motion in each kind of space is a screw motion, but the properties of these screw motions are somewhat different in the different spaces. Each kind of space admits of rotation as a special form of screw motion. Rotations in the three spaces are identical in their properties. Hyperbolic and elliptic space each have two types of translations. The properties of these translations are quite different. For a finite portion of space the elliptic, parabolic, and hyperbolic spaces are indistinguishable by their mechanical properties.

Professor Shaw's paper shows that the symbolic notation of Clebsch is simply an artificial form of a very natural notation for invariants, derived from quaternion expressions containing the operator

$$\nabla = i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}$$

in the case of three variables (*i. e.*, ternary quantics) and similar expressions in the general case of $n$-ary quantics.

The differential expressions considered by Dr. Snyder are analogous to those studied by Klein in line geometry. When

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2 \equiv 0,$$

$f(x_1, \cdots x_6) = 0$ represents the $\infty^3$ spheres which touch its point locus. When

$$\sum_{i=1}^{6}\left(\frac{\partial f}{\partial x_i}\right)^2 \sum_{i=1}^{6}\left(\frac{\partial \varphi}{\partial x_i}\right)^2 = \Sigma\left(\frac{\partial f}{\partial x_i}\right)\left(\frac{\partial \varphi}{\partial x_i}\right),$$

the spheres common to $f = 0$, $\varphi = 0$ are the principal spheres of their focal surface; if $f$ or $\varphi$ is linear this surface reduces to a curve. When the combinant of three complexes vanishes their common spheres envelope a curve, instead of an annular surface. Applied to the quadratic complex, the system of confocal cyclides with its focal curves and lines of curvature is obtained without the use of point coördinates.

Hermite's differential equation,

$$\frac{d^2y}{du^2} \equiv \left[n\,(n+1)\,p\,(u) + h\right] y$$

can be thrown by the substitution

$$x = p(u) \quad \text{or} \quad u = \int \frac{dx}{2\sqrt{f(x) = (x-e_1)\,(x-e_2)\,(x-e_3)}}$$

into the form

$$f(x)\frac{d^2y}{dx^2}+\tfrac{1}{2}f'(x)\frac{dy}{dx}-\frac{n(n+1)x+h}{4}y=0.$$

As is well known, it admits of two solutions whose product is a polynomial in $x$. Other differential equations having the same or an analogous property have been given by Fuchs, Brioschi, Markoff, Lindemann, and G. W. Hill. Professor E. B. Van Vleck's paper gives a simple method of finding all regular linear differential equations of the second order which admit of two solutions whose product is a polynomial. Besides the cases given by the above authors, numerous others are derived. Also the group properties which characterized the equation are considered. The second part of the paper is devoted to an investigation of the roots of the polynomial when the differential equation contains four singular points. The investigation is analogous to that undertaken by Klein for Hermite's equation, and contains the results of the latter as special case.

Professor Bolza presented a proof of the following theorem: If two elliptic integrals of the first kind

$$\int\frac{(xdx)}{\sqrt{(xa)\,(xa_1)\,(xa_2)\,(xa_3)}}\quad\text{and}\quad\int\frac{(ydy)}{\sqrt{(yb)\,(yb_1)\,(yb_2)\,(yb_3)}}$$

are transformable into each other by a rational cubic transformation which may, without loss of generality, always be supposed in the polar form: $y_1\dfrac{\partial F}{\partial x_1}+y_2\dfrac{\partial F}{\partial x_2}=0$, $F$ denoting a biquadratic in $x_1$, $x_2$, then *the roots* $a$, $a_1$, $a_2$, $a_3$ *and* $b$, $b_1$, $b_2$, $b_3$ *are the respective branch elements of two conjugate cubic involutions*, viz., the involution

$$\lambda_1\frac{\partial F}{\partial x_1}+\lambda_2\frac{\partial F}{\partial x_2}=0$$

and its conjugate.

Dr. Fields discusses elaborately the problem of the determination by a series of rational operations of the rational part of the general Abelian integral.

Dr. Hutchinson's paper on hyperelliptic functions is outlined by the following summary: Reduction of

$$\int F\left(z,\sqrt{f(z)}\right)dz\quad\left(f(z)=a_0z^6+a_1z^5+\cdots+a_6\right)$$

to sum of two elliptic integrals when $a_0, a_1, \cdots a_6$ satisfy the most general condition possible for a reduction involving a transformation of second degree. Study of a canonically dissected Riemann surface exhibiting the relation $\tau_{12} = \frac{1}{2}(\tau_{12} = \tau_{21}$ being moduli of periodicity) and its (1, 2) correspondence with each of the two elliptic Riemann surfaces. Reduction of Klein's covariant integral $Q_{xy}^{\nu\nu}$ to corresponding elliptic integrals. Reduction of Klein's hyperelliptic $\sigma$-functions to the Weierstrass elliptic $\sigma$-functions. Derivation of the Borchardt and Weber equations of the Kummer surface in terms of hyperelliptic $\sigma$-functions. In case of reducibility, the Kummer surface becomes either (1) a doubly covered quadric surface, or (2) a Fresnel wave surface, or (3) an elliptic Kummer surface of the "fourth order."

In Dr. McClintock's paper on the quintic equations a method is suggested for detecting the rational factors of reducible quintics, applicable also to equations of other degrees; also a method for recognizing quintics having just two imaginary roots, and therefore not resolvable. The author's former paper of 1885 (*Amer. Journal*, vol. 8), largely reproduced by Cayley (Collected Papers, end of vol. 4.), is recalled, including his new resolvent in $t$, or more generally in $\tau$, as well as his form in $v$ of the older (Malfatti's) resolvent; also his recognition and introduction of these rational quantities $t, v, s$, such that $s = t^2 v$, two of which are required in any resolution, and his formulæ for expressing the elements, $u_1, u_2, u_3, u_4$, and therefore the roots, in terms of $t, v$, and the coefficients. He now derives the same results (one of the formula being at the same time simplified) directly, without algebraic process, by a method applicable also to other equations, and precisely the reverse of that of Bezout and Euler, who define the elements as functions of the roots; he defines all his quantities, including the unknown quantity, as functions of the elements, being thus enabled to prove all his formulæ by immediate substitutions involving no irrational quantity. He next proves that all resolvable quintics depend upon four rational parameters, $q, r, t, w$, by assigning values to which resolvable quintics may be constructed at will, viz., the quintic being $y^5 + 10\gamma y^3 + 10\delta y^2 + 5\epsilon y + \xi = 0$, he makes $\gamma = ql$, $\delta = (r + t)\, l$, $v = l^2 \div (1 + w^2)$, where

$$l = [t^2 w^2 - r^2 (1 + w^2)] \div 4(q + 1)[q^2 (1 + w^2) - 1];$$

after which

$$e = \gamma^2 + v(3 + 4w) + (\gamma^3 - v)^{-1}(\gamma\delta^2 - \gamma t^2 v - 2\delta tvw),$$

$$\xi = (\gamma^3 - v)^{-1}(25tv^2 - \gamma t^2 v + \delta t^2 v - etv - 10\gamma^2 tv + \gamma^4 t - \gamma^2 et + \gamma\delta^2 t + 2\gamma\delta e - \delta^3).$$

He modifies this system for various critical cases, gives reasons why simpler parameters cannot be devised, notes that there are four quintics for which the value of $t$ is the same and that of $v$ is the same, and refers to earlier partial solutions, namely, for the trinomial $y^5 + 5ey + \xi = 0$, of the problem of assigning the necessary form of the coefficients of resolvable quintics. He proves finally, with comments, that if his sextic resolvent in $\tau$ be reduced to a quintic by dividing out the rational root, the resolvent in $\tau$ of the new quintic must have the same rational root.

Professor Maschke established the theorem that the coefficients of a linear homogeneous substitution group of finite period with any number of variables are cyclotomic, *i. e.*, are rationally expressible in terms of roots of unity.

Dr. Miller discusses certain properties of the commutator groups. The operation $sts^{-1}t^{-1}$ is said to be the commutator of the two operations $s$ and $t$. When $s$ and $t$ are commutative their commutator is identity and, *vice versa*. From the equation $sts^{-1}t^{-1} = (tst^{-1}s^{-1})^{-1}$ we see that the inverse of a commutator may be obtained by interchanging its two operators, and from the equations

$$l^{-1}sts^{-1}t^{-1}l = l^{-1}sts^{-1}ll^{-1}t^{-1}l = S_1tS_1^{-1}l^{-1}t^{-1}l = Sl^{-1}lS^{-1}l^{-1}t^{-1}l = STS^{-1}T^{-1}$$

it follows that the transforms of the commutator of $s$ and $t$ with respect to the operations of a group that includes these operators are commutators of the same group. The subgroup generated by all the commutators of a given group is self-conjugate. It is the smallest subgroup with respect to which the given group is isomorphic to a commutative group. When the given group is simple this subgroup is identical to it, but the converse is not true. The subgroup generated by the commutators of the given self-conjugate subgroup is also self-conjugate in the entire group. The necessary and sufficient condition that a group is solvable is that we arrive at identity by finding the subgroup generated by the commutators of a group, then the subgroup generated by the commutators of this subgroup, etc. If we let $s$ and $t$ represent successively all the operations of a group $sts^{-1}t^{-1} = 1$ $gk$ times, $g$ being the order of

the group and $k$ the number of its systems of conjugate operations. *References.*— *Quarterly Journal*, vol. 28, p. 266 (1896). Frobenius, *Berliner Sitzungsberichte*, December, 1896. Dedekind, *Mathematische Annalen*, January, 1897.

Collineations with invariant conic have eight free constants. The invariant condition determining the ninth has been found by seeking *Bahn-curven* belonging to iterations of a collineation, and specializing their parameters to reduce them to conics. Professor White attacks the problem directly, postulating at once an invariant conic which is not degenerate. After this a similar method gives corresponding conditions for invariant cubic. Both problems are elucidated and in part solved by geometric schemes.

Professor Moore's paper is a continuation of the author's investigations on triple systems. A triple system in $t$ letters is regular if it is invariant under a regular substitution group of order $t$ on its $t$ letters. An analysis is given of the general regular system and explicit construction of a system in $t = 6m + 3$ letters, regular with respect to any regular group of order $t = 6m + 3$, which contains a self-conjugate element of period 3, and a group of order $2m+1$ not containing that element. Professor Moore's paper appears in full in the present number of the BULLETIN.

Mr. Frankland's papers on the theory of discrete manifolds have been privately printed from time to time, the series now extending to Part IX. These papers contain a discussion of the notion of continuity, certain alternative implications of this notion as applied to space, the connection between the geometry of continuous manifolds and the tactical arrangement of units in a discrete manifold, "facts of nextness" and "microcycles or meshes or circuits" in a discrete manifold, hypotheses respecting the geometry of minute regions, suggesting the possibility of an explanation of mass, space occupancy and motion in terms of time and tactical arrangement only, discussion of certain principles which must underlie a suitable notation for theorems relating to discrete manifolds, and conditions under which the units of a discrete manifold constitute (a) an enumerable, and (b) a denumerable mass.

Mr. Philbrick's paper, which is to be published as a treatise, contains an elaborate discussion of a specially simple form of transition curve together with an historical note on this subject.

Dr. Martin presented a number of interesting arithmetical identities concerning sixth power numbers.

Professor Hathaway defines a function of one or more

complex number arguments which is linear and homogeneous in each argument and is reversed in sign by the interchange of any two of its arguments as a linear alternate or simply an alternate. The author proposes to investigate the geometric relations of quaternion and higher complex numbers, and to determine the relations that exist between alternate identities and integrations through a given space and over the boundary of that space. An instance of such a connection has been already discussed by the author in the Proceedings of the Indiana Academy of Science, 1891.

A portion of the Tuesday afternoon session was devoted to a general discussion of the following topics: (1) *The accurate definition of the subject matter of mathematics;* (2) *The vocabulary of mathematics, the possibility of correcting and enriching it by coöperative action.*

F. N. COLE.

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## CONCERNING REGULAR TRIPLE SYSTEMS.

BY PROFESSOR ELIAKIM HASTINGS MOORE.

Read before the AMERICAN MATHEMATICAL SOCIETY at its Fourth Summer Meeting, Toronto, Canada, August 17, 1897.

### § 1.

*Introduction. Definitions and notations.*

A $k$-ad $[l_1, \cdots, l_k]$
$k$-id $\{l_1, \cdots, l_k\}$

is an arrangement of $k$ letters or elements $l_1 \cdots l_k$ in which the order $\begin{matrix}\text{is}\\ \text{is not}\end{matrix}$ material. The letters are distinct.

A *triple system* $\Delta_t$ is an arrangement of $t$ letters in 3-adic triples in such a way that every 2-adic pair appears exactly once in some triple of the triple system. There is no question of order of the triples of the triple system. $t$ must have the form $t = 6m + 1$ or $t = 6m + 3$; $t$ denotes always a number of such (say) *triple form.*

A triple system $\Delta_t$ is invariant under a certain (largest) substitution group $G^t$ on its $t$ elements; the $\Delta_t$ and the $G^t$ belong each to the other; the $G^t$ is a *triple group.*

In the definitive theory of triple systems and triple groups we must, among other things, determine whether a (any) particular group $G^t$ of triple degree $t$ is a triple group, and, if so, we must construct its various triple systems $\Delta_t$.

A system $\Delta_t$ is *transitive* if its group $G^t$ is transitive.

A transitive system $\Delta_t$ is *regular* if its group $G^t$ contains a regular subgroup $H_t^t$ of order $t$ on the $t$ elements of the $\Delta_t$; the group $H_t^t$ is a *group of regularity* of the *system* $\Delta_t$; the system $\Delta_t$ has with respect to the group $H_t^t$ the *regular aspect* $\Delta_t \mid H_t^t$. A regular group $H_t^t$ is uniquely determined by the corresponding *abstract group* $H_t$.

Mr. Netto has exhibited, cyclic, or, as I say, *cyclic*-regular $\Delta_t(1°)$ for $t = 6m + 1 = p$ ($p$ any prime) and $(2°)$ for $t = 6m + 3 = 3p$ ($p$ any prime of the form $p = 6k + 5$).

Mr. Heffter in the current number (vol. 49, part 1) of the *Mathematische Annalen*, has brought out clearly the "difference problems" underlying the problems of construction of the general cyclic-regular $\Delta_t$ for $t=6m+1, 6m+3$. He then exhibits cyclic-regular $\Delta_t(3°)$ for $t=6m+1=12k+7 = 3p - 2$ ($p$ any prime of the form $p = 4k + 3$ with the primitive root 2) and $(4°)$ (by a slight modification of Mr. Netto's $(2°)$ system) for $t = 6m + 3 = 3p$ ($p$ any prime greater than 3).

Now a *cyclic* group $H_t^t$ is the *simplest Abelian* or commutative group $H_t$.

In a paper, *Concerning Abelian-Regular Transitive Triple Systems*, forwarded to the *Annalen* three weeks ago, I have analyzed (*l. c.*, § 1) the general Abelian-regular $\Delta_t$, and have exhibited (*l. c.*, (*a*) § 2, (*b*) §§ 3, 4) Abelian-regular $\Delta_t$ with respect to the abstract Abelian group $H_t$ where

(*a*) $t$ is any integer $t = 6m + 3$,

$H_t$ is any Abelian group of order $t$ having one invariant 3; or

(*b*) $t$ is any integer $t = 6m + 1$ in which every prime factor of the form $p = 6k + 5$ enters an even number of times;

$H_t$ is any Abelian group of order $t$ in whose invariant-character every such prime enters always with the exponent 1.

The systems (*a*, *b*) are sweeping generalizations of the systems $(2°, 4°; 1°)$ of Messrs. Netto and Heffter.

My analysis of the general Abelian-regular system $\Delta_t$ was so phrased as to admit of immediate application in the constructions (*a*, *b*). In this paper the general *regular triple system* $\Delta_t$ is subjected to a similar analysis (§ 2), on the

basis of which a generalization of the system ($a$) is effected (§ 4).

## § 2.

*Regular triple systems* $\Delta_t \mid H_t'$ *and the corresponding sextette separations* $\sigma_{m_1, m_2} \mid H_t'$.

We denote the $t$ elements of the group $H_t$ and the $t$ elements of the regular system $\Delta_t \mid H_t'$ without confusion by the same notation—$A_1, \cdots, A_t$. The substitution group $H_t'$ is made up of the substitutions

$$(1) \qquad A_j = \begin{pmatrix} A_1, \cdots\cdots, A_i, \cdots\cdots, A_t \\ A_1A_j, \cdots, A_iA_j, \cdots, A_tA_j \end{pmatrix} \qquad (j = 1, 2, \cdots, t).$$

The regular system $\Delta_t \mid H_t'$ is invariant under the $t$ substitutions $A_j$ (1) of the $H_t'$. Hence the $\Delta_t$ contains with the triple $[A_{i_1} A_{i_2} A_{i_3}]$ the $t$ triples (not necessarily all distinct)

$$(2) \qquad [A_{i_1} A_j, A_{i_2} A_j, A_{i_3} A_j] \qquad (j=1, 2, \cdots, t).$$

Writing the triple or 3-ad $[A_{i_1} A_{i_2} A_{i_3}]$ having regard to the order of the letters as a 3-id $\{A_{i_1} A_{i_2} A_{i_3}\}$ we see that the corresponding sextette

$$(3) \qquad \sigma\{A_{i_1} A_{i_2} A_{i_3}\} = \begin{Bmatrix} A_{i_2} A_{i_3}^{-1}, & A_{i_3} A_{i_1}^{-1}, & A_{i_1} A_{i_2}^{-1} \\ A_{i_3} A_{i_2}^{-1}, & A_{i_1} A_{i_3}^{-1}, & A_{i_2} A_{i_1}^{-1} \end{Bmatrix}$$

is an invariant and indeed a characteristic invariant for the 3-idic triples

$$\{A_{i_1} A_j, A_{i_2} A_j, A_{i_3} A_j\}$$

of this set (2).

Such a sextette $\sigma\{A_{i_1} A_{i_2} A_{i_3}\}$ has the necessary and sufficient form

$$(4) \qquad \sigma \mid H_t = \begin{Bmatrix} B_1, & B_2, & B_3 \\ B_1^{-1}, & B_2^{-1}, & B_3^{-1} \end{Bmatrix}$$

where*

$$(5) \qquad B_1 B_2 B_3 = I, \quad B_i \neq I, \quad B_i \neq B_j^{-1} \quad (i, j = 1, 2, 3),$$

* The group $H_t$ is of odd order $t$. Every triple has three distinct letters. Two triples having two letters in common have also the third of each in common. The sextette $\sigma \mid H_t$ with $B_1 B_2 B_3 = I$ belongs to the 3-idic triples

$$(I, B_3^{-1}, B_2), \quad (B_3, I, B_1^{-1}), \quad (B_2^{-1}, B_1, I).$$

From these remarks one draws conclusions (5, 6) of the text.

the $B$'s being elements and the $I$ the identity element of the group $H_t$. There are in all two types of sextettes $\sigma \mid H_t$:

$$(6) \qquad \begin{array}{l} (1°)\ B_1 B_2 B_3 \text{ are distinct;} \\ (2°)\ B_1 = B_2 = B_3. \end{array}$$

According as the sextette $\sigma \mid H_t$ is $\sigma_1 \mid H_t$ or $\sigma_2 \mid H_t$ of the type 1° or 2°, it contains six or two distinct elements and the corresponding set of triples (2) contains $t$ or $\frac{1}{3}t$ triples

and is indeed a (tactical) configuration* $Cf_1 \begin{pmatrix} t & 3 \\ 3 & t \end{pmatrix}$

or $Cf_2 \begin{pmatrix} t & 1 \\ 3 & \frac{1}{3}t \end{pmatrix}$ regular with respect to the group $H_t'$ of substitutions $A_j$ (1). The type 2° occurs only if $t$ has the form $t = 6m + 3$.

The system $\Delta_t \mid H_t'$ is the composition of $m_1$ configurations $Cf_1 \mid H_t'$ of type 1° and $m_2$ configurations $Cf_2 \mid H_t'$ of type 2°, with distinct triples. Here $tm_1 + \frac{1}{2}tm_2 = \frac{1}{6}t(t-1)$, $\therefore 3m_1 + m_2 = \frac{1}{2}(t-1) = 3m$ or $3m+1$. Hence we have

$$(7) \qquad t = \begin{matrix} 6m+1 \\ 6m+3 \end{matrix}, \quad (m_1, m_2) = \begin{matrix} (m, 0) \\ (m - m', 1 + 3m') \end{matrix}$$
$$(0 \leqq m' \leqq m)$$

Corresponding to and characteristic of this *configuration-separation* $Cf_{m_1, m_2} \mid H_t'$ of the system $\Delta_t \mid H_t'$ is a *sextette-separation* $\sigma_{m_1, m_2} \mid H_t$ of the $t-1$ elements $A$ $(A + I)$ into $m_1$ sextettes $\sigma_1 \mid H_t$ and $m_2$ sextettes $\sigma_2 \mid H_t$ (in which repetitions of elements occur only within the individual sextettes $\sigma_2 \mid H_t$).

Conversely, with respect to any abstract group $H_t$ of order $t = 6m+1$, $6m+3$ any such sextette-separation $\sigma_{m_1, m_2} \mid H_t$ of the $t-1$ elements $A$ $(A + I)$ serves uniquely to define a regular triple system $\Delta_t \mid H_t'$.

We consider as essentially the same the six sextettes $\sigma_1 \mid H_t$ derived from the two

* I use the general matrix notation for configurations introduced in I (The General Tactical Configuration: Definition and Notation) of my paper *Tactical Memoranda* I–III (*American Journal of Mathematics*, vol. 18, pp. 264–303, 1896).

$$(8) \qquad \left\{ \begin{matrix} B_1 , & B_2 , & B_3 \\ B_2^{-1}, & B_3^{-1}, & B_1^{-1} \end{matrix} \right\} , \qquad \left\{ \begin{matrix} B_1^{-1}, & B_3^{-1}, & B_2^{-1} \\ B_1 , & B_3 , & B_2 \end{matrix} \right\}$$

by cyclical permutation of their columns, since they arise from the six 3-idic triples corresponding to one 3-adic triple.

If in a sextette $\sigma_1 \mid H_t$ (4) we interchange

$$B_i, B_i^{-1} \ (i = 1, 2, 3),$$

the new say *reciprocal* sextette is a $\sigma_1 \mid H_t$ if and only if $B_1^{-1} B_2^{-1} B_3^{-1} = I$. This happens, for instance, always if $H_t$ is an Abelian group. Two reciprocal sextettes of type 1° are essentially distinct.

## § 3

*Explicit exhibition of a sextette-separation* $\sigma_{0,\,1+3m} \mid H_{t=3^k=6m+3}$, *where* $H_t$ *is any group of order* $t = 3^k = 6m + 3$ *whose elements not the identity are all of period* 3.

Corresponding to the $1 + 3m$ pairs of reciprocal elements $A \ (A \neq I)$ of the group $H_t$ we have the separation of those elements (of period 3) into $1 + 3m$ sextettes $\sigma_2 \mid H_t$ of type 2°.

For the (cyclid) Abelian $H_{t=3^k}$ generated by $k$ generators each of period 3 this separation underlies the Abelian-regular $\Delta_3^k \mid H_{3^k}^{3^k}$ whose group is the linear group modulo 3. (Netto, *Substitutionentheorie*, pp. 224–234.)

## § 4

*Explicit exhibition of* $2^m$ *sextette-separations* $\sigma_{m,1} \mid H_{t=6m+3}$, *where* $H_t$ *is any group of order* $t=6m+3$ *having a self-conjugate element* $A_0$ *of period* 3 *and a subgroup* $K_{2m+1}$ *of order* $2m+1$ *not containing* $A_0$.

$C$ denotes always an element of the subgroup $K_{2m+1}$. The $2m$ elements $C(C \neq I)$ separate uniquely into $m$ pairs of reciprocal elements $C, C^{-1}$. If the $C_i (i = 1, \cdots, m)$ form a system of representatives of these $m$ pairs, then so do the $C_i^2 (i = 1, \cdots, m)$.

The group $K_{2m+1}$ extends by $A_0$ to the group $H_{t=6m+3}$. Since $A_0$ is a self-conjugate element in $H_t$, it is commutative with every element $C$. The elements $A$ of $H_t$ have the form $C$, $CA_0$, $CA_0^2$.

The separation $\sigma_{m,1} \mid H_t$ of the $t - 1 = 6m + 2$ elements $A(C \neq I)$ of the $H_t$ consists of the one sextette

$$(1) \qquad \sigma_3 \mid H_t = \left\{ \begin{matrix} A_0, A_0, A_0 \\ A_0^2, A_0^2, A_0^2 \end{matrix} \right\} \qquad (A_0^3 = I)$$

and of the $m$ sextettes $\sigma_1^{(C_i)} \mid H_t$ depending upon a representative system $C_i = C_1, \cdots, C_m$ of the $m$ pairs of reciprocal elements $C(C + I)$,

$$(2) \qquad \sigma_1^{(C)} \mid H_t = \left\{ \begin{matrix} CA_0, & CA_0^2, & C^{-2} \\ C^{-1}A_0^2, & C^{-1}A_0, & C^2 \end{matrix} \right\},$$

or

$$(3) \qquad \sigma_1^{(C)} \mid H_t = \left\{ \begin{matrix} CA_0^2, & CA_0, & C^{-2} \\ C^{-1}A_0, & C^{-1}A_0^2, & C^2 \end{matrix} \right\}.$$

The two sextettes $\sigma_1^{(C)}$, $\sigma_1^{(C^{-1})}$ of (2) or (3) are essentially the same, while $\sigma_1^{(C)}$ (2) and $\sigma_1^{(C)}$ (3) are essentially distinct (reciprocals).

According to the choice of (2) or (3) for each pair $C$, $C^{-1}$ we have in all $2^m$ sextette-separations $\sigma_{m,1} \mid H_t$ and so $2^m$ regular $\Delta_t \mid H_t'$ for every abstract group $H_t$ of the character in question. In particular, since every Abelian group $H_t$ with one invariant 3 is such a group $H_t$, we have the Abelian-regular $\Delta_t$ $H_t$ (§ 1, *a*).

For still more general types of groups $H_t$ of order $t = 6m + 3$ we may by suitable modification of the preceding process exhibit a sextette-separation $\sigma_{m,1} \mid H_t$. Thus, for example, for those $H_t$ with the following properties: (1) the $H_t$ has an element $A_0$ of period 3, (2) the group $I$, $A_0$, $A_0^2$ extends to the $H_t$ by the identity and certain $2m$ extenders $C$ by pairs reciprocal, (3) a certain representative system $C_i (i = 1, \cdots, m)$ of these $m$ pairs is invariant under transformation by $A_0$, (4) the system $C_i A_0 C_i A_0^2 (i = 1, \cdots, m)$ is a representative-system of these $m$ pairs. This type of group $H_t$ contains the type previously discussed and also, for instance, the group $H_{21}$ generated by two generators $A_0$, $C$ subject to the generational relations

$$A_0^3 = I, \; C^7 = I, A_0 C = C^2 A_0.$$

THE UNIVERSITY OF CHICAGO, *May* 4, 1897.

# COLLINEATIONS IN A PLANE WITH INVARIANT QUADRIC OR CUBIC CURVES.

BY PROFESSOR H. S. WHITE.

(Read before the American Mathematical Society at its Fourth Summer Meeting, Toronto, Canada, August 17, 1897.)

A LINEAR substitution applied to a ternary form will change, in general, the ratios among its coefficients. Unless the substitution is a specialized one, there are three independent linear forms which it leaves unchanged save by a multiplicative constant, but no such forms of higher order except those that are reducible to products of these three linear factors. So much being premised, it is apparent that if any irreducible forms of higher order are unchanged by a particular linear substitution, it must be by reason of some relation among the coefficients of that substitution; and further, that such a relation must be unaltered when this first substitution is transformed by (*not* compounded with) a second. Such relations are expressible in fact by equations involving only the invariants of the first linear substitution. The expression of a conditional relation in invariant form when there is a quadric invariant of the substitution has been effected by integrating a differential equation to determine transcendental invariant forms, then discussing what relations among the parameters will reduce those forms to quadrics. This method is indeed exhaustive; but for the special problem an exhaustive method is not indispensable. It is possible to obtain the invariant conditional equation by quite elementary processes, not only when quadric forms are to be left unchanged, but also when beside the quadrics there are proper cubic invariants. As preliminary to this main theme I will restate well-known formulæ concerning linear invariants and fundamental invariants of a linear substitution or collineation.

A ternary linear substitution or collineation may be represented by the equation (in Clebsch-Aronhold notation):

$$a_x u_a = 0,$$

the equation in line coördinates $(u)$ of the point into which any point $(x)$ is transformed. More explicitly it is written:

$$y_1 = a_1\alpha_1 x_1 + a_2\alpha_1 x_2 + a_3\alpha_1 x_3$$
$$y_2 = a_1\alpha_2 x_1 + a_2\alpha_2 x_2 + a_3\alpha_2 x_3$$
$$y_3 = a_1\alpha_3 x_1 + a_2\alpha_3 x_2 + a_3\alpha_3 x_3.$$

Any invariant of the mixed form or connex $a_\alpha u_\alpha$ is called an invariant of the collineation. There are three independent invariants, represented by $i_1, i_2, i_3$;*

$$i_1 = a_\alpha\,;\quad i_2 = a_\beta b_\alpha\,;\quad i_3 = a_\beta b_\gamma c_\alpha.$$

The three linear forms unaltered by the collineation are determined by the aid of the characteristic equation. If the equation of one invariant line were $A_x = 0$, and that of its transformed equivalent $A_y = 0$, since the two can differ only by a multiplicative constant $\lambda$, we must have identically

$$A_y = A_\alpha a_x \equiv \lambda A_x.$$

Separating this into three, and eliminating the $(A)$, we find the characteristic equation:

$$^* \begin{vmatrix} a_1\alpha_1 - \lambda & a_1\alpha_2 & a_1\alpha_3 \\ a_2\alpha_1 & a_2\alpha_2 - \lambda & a_2\alpha_3 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 - \lambda \end{vmatrix} = 0\,;$$

or reduced

$$\lambda^3 - i_1\lambda^2 + \tfrac{1}{2}(i_1^2 - i_2)\lambda - \tfrac{1}{6}(i_1^3 - 3i_1i_2 + 2i_3) = 0.$$

If the three roots are $\lambda_1, \lambda_2, \lambda_3$, the corresponding linear forms are found by solving the identical equation for $A_x$; or by factoring the bordered determinant:

$$\begin{vmatrix} a_1\alpha_1 - \lambda_i & a_1\alpha_2 & a_1\alpha_3 & u_1 \\ a_2\alpha_1 & a_2\alpha_2 - \lambda_i & a_2\alpha_3 & u_2 \\ a_3\alpha_1 & a_3\alpha_2 & a_3\alpha_3 - \lambda_i & u_3 \\ x_1 & x_2 & x_3 & 0 \end{vmatrix} = \varphi_i(x) \cdot \psi_i(u)$$

Excluding arbitrarily the case of degenerate collineation, where the absolute term in the characteristic equation would be zero, we have corresponding to the three roots three linearly independent forms $\varphi_1(x)$, $\varphi_2(x)$, $\varphi_3(x)$, and

* Clebsch and Gordan use the designations $i, i_1, i_2$, respectively. I adopt the subscript showing their degrees in the coefficients of the collineation.

may adopt them as independent variables $x_1, x_2, x_3$. The connex then has the simplified expression:

$$u_\alpha a_\alpha = pu_1x_1 + qu_2x_2 + ru_3x_3.$$

We shall assume also that the quantities $p, q, r$ are all different. These are obviously the roots of the characteristic equation, so that we have the relations:

$$\begin{aligned} p + q + r &= i_1, \\ qr + rp + pq &= \tfrac{1}{2}(i_1^2 - i_2), \\ pqr &= \tfrac{1}{6}(i_1^3 - 3i_1i_2 + 2i_3). \end{aligned}$$

The three invariants are seen to be the sums of first, second and third powers respectively of these three roots $p, q, r$.

That a quadric $A_x^2$ may be transformed into itself by the collineation $u_\alpha a_\alpha = 0$, the condition must be identically fulfilled

$$A_\alpha A_\beta a_\alpha b_\alpha \equiv \lambda A_x^2.$$

Resolving this identity into six equations and eliminating the unknown coefficients $A_\alpha$, we have the characteristic equation for the factor $\lambda$:

$$(p^2 - \lambda)\,(q^2 - \lambda)\,(r^2 - \lambda)\,(qr - \lambda)\,(rp - \lambda)\,(pq - \lambda) = 0.$$

If the roots of this equation are all different, there are six determinate quadric forms $A_x^2$, and no more. These six, however, are obviously nothing more than products of the invariant linear fo ms. For example,

$$\begin{aligned} &\text{to } \lambda = p^2 \text{ corresponds } A_x^2 = x_1^2, \\ &\text{to } \lambda = pr \text{ corresponds } A_x^2 = x_1x_3, \text{ etc.} \end{aligned}$$

How then can any irreducible quadric occur with the invariant property? Certainly only when two roots of this characteristic equation are equal. In that case there will be not only one, but a simple infinity of quadric forms satisfying the condition. Of possible equalities among the roots, the following comprise all types:

$$\begin{aligned} &(1)\ p^2 = q^2, \qquad (3)\ pq = qr, \\ &(2)\ p^2 = pr, \qquad (4)\ pq = r^2. \end{aligned}$$

These we will consider in their order.

1. Corresponding to $p^3 = q^2$ we should find the system of quadrics with arbitrary parameter $m$:

$$x_1^2 - mx_2^2,$$

all of which are reducible.

2 and 3. If $p^2 = pr$, or $pq = qr$, it follows that $p = r$, and the corresponding quadrics are the two systems, all reducible:

$$Ax_1^2 + Bx_1x_3 + Cx_3^2,$$
$$Dx_1x_2 + Ex_2x_3$$

4. Accordingly if any non-singular collineation can leave invariant any irreducible quadric, it must satisfy a condition of the type $pq = r^2$. The quadrics corresponding contain one arbitrary parameter $m$:

$$x_1x_2 - mx_3^2.$$

These are certainly irreducible except for $m = 0$ or $m = \infty$. Rationalizing now this typical condition $pq = r^2$ in terms of the three invariants, we find

$$pqr = \left(\frac{pq + qr + rp}{p + q + r}\right)^3, \text{ or}$$

$$3(i_1^2 - i_2)^3 - 4i_1^3(i_1^3 - 3i_1i_2 + 2i_3) = 0.*$$

Since also the argument is reversible, the result may be formulated thus:

*If any proper conic is left unaltered by a non-singular collineation of the plane, then every conic of a simply infinite sheaf must share the invariant property; and the necessary and sufficient condition for the occurrence of such invariable conics is the following relation among the three rational invariants of the collineation:*

$$3(i_1^2 - i_2)^3 - 4i_1^3(i_1^3 - 3i_1i_2 + 2i_3) = 0.$$

Before taking up cubic curves, it is useful to review a part of the above argument in geometric language. Consider as before a collineation that leaves unaltered the three lines of a non-vanishing triangle. If also some proper conic is to be transformed into itself, then the poles of these three

* The coefficient 4 of the second term of this condition equation is erroneously omitted in Clebsch-Lindeman, Vorlesungen über Geometrie, I., p. 994.

lines with respect to the conic must be invariant points. Such points cannot lie on a line, they must be therefore the three vertices of the invariant triangle. Further, this self-polar triangle must have two sides tangent to the conic; for if none were tangent, the six points in which they cut the conic would need to lie in involution, whence their three join-lines must meet in one point, contrary to our hypothesis. One side accordingly must be tangent to the conic, and since the triangle is self-polar a second side also is tangent, while the third is the chord of contact. The conic referred to this triangle has the equation:

$$Ax_1x_2 + Bx_3^2 = 0$$

while the collineation is of the form:

$$y_1 = px_1, \quad y_2 = qx_2, \quad y_3 = rx_3.$$

Substituting in the equation of the conic, we find as before the condition for its invariance,

$$pq = r^2,$$

and this gives the same relation among the invariants.

The collineation has three invariants, and if two conditions are imposed upon these, they determine its *absolute invariants*; no more conditions can be imposed without specializing the invariant triangle. We know that there are general collineations transforming a plane cubic into itself. Let it be required to determine their two invariant characteristics. If a proper non-singular cubic is transformed into itself, so are also necessarily both the conic polars and the linear polars of the three fixed points of the collineation. Two conic polars can never be doubly tangent while their poles are distinct, and so cannot belong to any such invariant sheaf as that discussed above. There must be three different sheaves of invariant conics, one in each sheaf being the conic polar of one vertex of the fixed triangle, and touching two sides where they are met by the third. Now two sorts of coördination are possible; either each vertex lies on its conic polar and on the cubic, or else each is the pole of the opposite conic and line; the other two hypotheses are excluded by the order of the cubic. Of the two possibilities the former covers two distinct coördinations of pole and polar, the latter of course but one.

For both cases alike, since an invariant conic occurs in

each of the three possible systems, the three conditions must be fulfilled:

$$qr = p^2, \quad rp = q^2, \quad pq = r^2.$$

Of these only two are independent. From them result the three sets of equalities of third degree:

(1) $$p^3q = q^3r = r^3p,$$

(2) $$pq^3 = qr^3 = rp^3,$$

(3) $$p^3 = q^3 = r^3 = pqr.$$

The equivalent relation among the invariants is:

$$i_1 = 0, \quad i_2 = 0, \quad i_3 + 0 \quad (i_3 = 3p^3).$$

Corresponding to the three sets of equal roots of the characteristic equation for invariant cubics, there are three systems having the following equations:

(1) $$Ax_1^2x_2 + Bx_2^2x_3 + Cx_3^2x_1 = 0,$$

(2) $$Ax_1x_2^2 + Bx_2x_3^2 + Cx_3x_1^2 = 0,$$

(3) $$Ax_1^3 + Bx_2^3 + Cx_3^3 + Dx_1x_2x_3 = 0.$$

For each of the first two systems the triangle is simultaneously inscribed in and circumscribed about each cubic of the system, and these cubics are of the special kind called *æquianharmonic*.* For every cubic of the third system the triangle consists of three lines of inflexion. This result is recapitulated in the statement:

*In order that a non-singular collineation may leave unaltered a non-singular plane cubic curve, it is necessary and sufficient that its invariants of first and second degree should vanish, while its invariant of third degree remains different from zero; and then three discrete systems of cubics enjoy the invariant property, the one containing a threefold infinity of general cubics, the other two containing each a twofold infinity of æquianharmonic cubics. All of the former system have their inflexions upon the fixed triangle, all of the latter systems are simultaneously inscribed in and circumscribed about the fixed triangle.*

It will be remembered that every non-singular cubic has eight collineations into itself, besides the identity, as well

* Clebsch-Lindemann, Geometrie, I., p. 579.

as nine others of a sort excluded from the present consideration, namely perspective transformations. It is of interest to observe that the three other inflexional triangles of each invariant cubic of the third class are themselves members of the system, and are therefore transformed into themselves, not interchanged, by the eight non-perspective collineations of the cubic.

Of plane curves of higher order than the third, it is easily shown that only highly specialized classes are collinear with themselves, and that of these classes the groups are correspondingly small. It appears possible, however, to extend this method of inquiry to such interesting topics as these:

(1) What simultaneous invariant conditions must two non-singular collineations satisfy in order to belong to the group leaving a common conic unaltered?

(2) What invariant conditions are met by collineations which leave unaltered a quadric surface? a twisted cubic? a twisted quartic curve?

NORTHWESTERN UNIVERSITY,
*August*, 1897.

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## A GENERATING FUNCTION FOR THE NUMBER OF PERMUTATIONS WITH AN ASSIGNED NUMBER OF SEQUENCES.

BY PROFESSOR F. MORLEY.

(Read before the American Mathematical Society at the Meeting of May 29, 1897.)

§ 1. *André's Recurrence-formula.* In *Liouville's Journal*, 1895, and in earlier memoirs there referred to, M. André proves the formula

$$P_{n,s} = sP_{n-1,s} + 2P_{n-1,s-1} + (n-s)P_{n-1,s-2} \qquad (1)$$

where $P_{n,s}$ is the number of permutations of $n$ things (say of the numbers $1, 2, \cdots n$) with $s$ sequences; and shows that (taking the number of sequences as great as possible) the numbers $\frac{1}{2}P_{n+1,n}$ are the coefficients of $x^n/n!$ in $1/(1-\sin x)$, when expressed as a Maclaurin series.

My object is to obtain a function of $x$ and $y$ which when developed in positive integer powers of $x$ and $y$ will have $P_{n,s}$ as the general coefficient.

It is convenient to replace the consideration of sequences by that of *runs*, where I define a run as three adjacent numbers in order of magnitude (ascending or descending). If there are $m$ numbers in order of magnitude they give $m-2$ runs, but one sequence. Whenever, by rearranging the numbers, we gain a run, we lose a sequence, so that if in a permutation of $n+1$ things there are $r$ runs and $s$ sequences then

$$r+s=\text{constant};$$

or since in the natural order there are $n-1$ runs and one sequence

$$r+s=n.$$

Let $c_{r,n}$ be half the number of permutations of the $n+1$ things, when there are $r$ runs; that $c_{r,n}$ is an integer is clear because any permutation when read backwards yields another with the same number of runs. We have then, rewriting André's formula

$$c_{r,n}=(n-r)\,c_{r-1,n-1}+2c_{r,n-1}+(r+1)\,c_{r+1,n-1}. \qquad (2)$$

The initial values are known from the definition, to be $c_{0,1}=1$, $c_{0,2}=2$, $c_{1,2}=1$; and so on; whence using for the higher values the recurrence formula we have the table:

| $r=$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $n=1$ | 1 | | | | |
| 2 | 2 | 1 | | | |
| 3 | 5 | 6 | 1 | | |
| 4 | 16 | 29 | 14 | 1 | |
| 5 | 61 | 150 | 118 | 30 | 1 |

It must be noticed that in accordance with this table we have $c_{-1,n}=0$ for any positive $n$.

§ 2. *The differential equation.* There is, of course, freedom in the choice of a double Maclaurin series which shall have as its cofficients the numbers $c_{r,n}$. Thus we can take as the typical term

$$c_{r,n}x^r y^n,\quad c_{r,n}x^r y^n/r!,\quad c_{r,n}x^r y^n/n!,\quad \text{or } c_{r,n}x^r y^n/r!\,n!$$

Of these the third turns out to be the most convenient. Let then

$$z = \sum_{r=0}^{\infty} \sum_{n=1}^{\infty} c_{r,n} x^r y^n / n ! \qquad (3)$$

Then

$$D_y z = \sum_{r=0} \sum_{n=1} c_{r,n} x^r y^{n-1} / (n-1)!$$

or from (2)

$$D_y z = \Sigma\Sigma [2c_{r,n-1} + (r+1)c_{r+1,n-1}$$
$$+ (n-r)c_{r-1,n-1}] x^r y^{n-1} / (n-1)!$$
$$= 2\sum_{r=0} \sum_{n=0} c_{r,n} x^r y^n / n! + \sum_{r=1} \sum_{n=0} c_{r,n} r x^{r-1} y^n / n!$$
$$- \sum_{r=-1} \sum_{n=0} c_{r,n} r x^{r+1} y^n / n! + \sum_{r=-1} \sum_{n=0} c_{r,n} x^{r+1} y^n / (n-1)!$$
$$= 2(z + Px) + D_x(z + Px) - x^2 D_x(z + Px) + xyD_y(z + Px),$$

where $Px$ is the series $\sum_{r=0}^{\infty} c_{r,0} x^r$. We have then, if $z_1 = z + Px$, the equation

$$(1 - xy) D_y z_1 - (1 - x^2) D_x z_1 = 2z_1. \qquad (4)$$

§ 3. *Solution of the differential equation.* We have to solve

$$\frac{dy}{1 - xy} = -\frac{dx}{1 - x^2} = \frac{dz_1}{2z_1}.$$

From the second and third expressions

$$z_1 \frac{1 + x}{1 - x} = \text{a constant } a.$$

To solve $\dfrac{dy}{1 - xy} = -\dfrac{dx}{1 - x^2}$ write $x = \sin t$.

Then $$dy \cos t = -(1 - y \sin t) dt$$

or $$y \cos t + t = b.$$

Hence the solution of (4) is

$$z_1 \frac{1 + x}{1 - x} = f(y \cos t + t).$$

Now André has proved that when $x = 0$,

$$z = \frac{1}{1 - \sin y} - 1$$

Hence the form of the function $f$ is known; and we have

$$(z + Px)\frac{1+x}{1-x} = \frac{1}{1 - \sin(y\cos t + t)}.$$

We determine $Px$ by the condition that $z = 0$ when $y = 0$. This gives

$$Px\frac{1+x}{1-x} = \frac{1}{1-\sin t} = \frac{1}{1-x}.$$

Hence finally the generating function is

$$z = \frac{1-x}{1+x}\,\frac{1}{1-\sin(y\cos t + t)} - \frac{1}{1+x} \qquad (5)$$

where $t$ is that arc whose sine is $x$ which vanishes with $x$.

As a test of correctness let $x = 1$. The series (3) is then

$$z = \tfrac{1}{2}(2!\ y + 3!\ y^2/2! + 4!\ y^3/3! + \cdots)$$

$$= \tfrac{1}{2}\left(\frac{1}{(1-y)^2} - 1\right).$$

And the limit for the function (5) is

$$\tfrac{1}{2}\left(\lim_{t=\pi/2}\frac{1-\sin t}{1-\sin(y\cos t + t)} - 1\right)$$

$$= \tfrac{1}{2}\left(\lim_{\epsilon=0}\frac{1-\cos\epsilon}{1-\cos(\epsilon - y\sin\epsilon)} - 1\right)$$

$$= \tfrac{1}{2}\left(\frac{1}{(1-y)^2} - 1\right).$$

§ 4. *The polynomials in $x$.* As an application of (5), we can expand directly in powers of $y$ and thus examine the polynomials $Q_n x$ which is the coefficient of $y^n/n!$, this polynomial being of degree $n$.

We have

$$Q_n x = \frac{1-x}{1+x} \lim_{y=0} D_y^n \frac{1}{1-\sin(y\cos t + t)}$$

$$= \frac{1-x}{1+x} \cos^n t \, D_t^n \frac{1}{1-\sin t} \qquad (6)$$

From this form the polynomials are readily investigated. It may be remarked that, since

$$\frac{1}{1-\sin t} = \frac{1}{1-\exp i(t-\pi/2)} - \left(\frac{1}{1-\exp i(t-\pi/2)}\right)^2,$$

the explicit determination of

$$D_t^n \frac{1}{1-\sin t}$$

reduces to that of

$$D_u^n \frac{1}{1-e^u}.$$

Let this expression be

$$\frac{A_1}{1-e^u} + \frac{A_2}{(1-e^u)^2} + \cdots + \frac{A_{n+1}}{(1-e^u)^{n+1}};$$

then considering only negative powers of $u$ we have

$$Pu + (-)^{n+1} n!/u^{n+1} \equiv \frac{A_1}{1-e^u} + \cdots + \frac{A_{n+1}}{(1-e^u)^{n+1}},$$

whence the coefficients $A$ are to be found; and writin $1-e^u = v$ we have

$$\frac{A_{n+1}}{v^{n+1}} + \frac{A_n}{v^n} + \cdots + \frac{A_1}{v} \equiv \frac{(-)^{n+1} n!}{(\log(1-v))^{n+1}} + Pv,$$

whence, so far as the $n^{\text{th}}$ power of $v$,

$$A_{n+1} + A_n v + \cdots + A_1 v^n \equiv \frac{n!}{(1 + v/2 + v^2/3 + \cdots)^{n+1}},$$

that is the coefficients $A$ are determined as the coefficients of the first $n$ powers in

$$n!(1 + v/2 + v^2/3 + \cdots)^{-n-1}$$

Formula (6) shows at once that when $n$ is even (say $n = 2m$), $Q_n$ is exactly divisible by $(1+x)^{m-1}$. For the numerator has the factor $(1-x^2)^m$, and the denominator is $(1+x)(1-x)^{2m+1}$. Similarly when $n = 2m-1$, $Q_n$ is divisible by $(1+x)^{m-1}$; for $D_t^n \frac{1}{1-\sin t}$ has now the factor $\cos t$ in the numerator, whence $Q_n$ has still in its numerator the factor $(1-x^2)^m$. For example, from the table

$$Q_5 x = 61 + 150x + 118x^2 + 30x^3 + x^4$$

and this is exactly divisible by $(1+x)^2$.

HAVERFORD COLLEGE,
*June*, 1897.

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## LINE GEOMETRY.

*La Géométrie réglée et ses applications.* Par G. KOENIGS. Paris, Gauthier-Villars et Fils. 1895. 4to. 146 pp.

The first general impulse to the study of the geometry of the straight line was given by Plücker's posthumous work in 1868–9, which was soon followed by a large number of contributions from German, French and Italian mathematicians; the English people were soon interested in the application of the new geometry to mechanics and physics.

In the work of Plücker, point and plane coördinates are used nearly throughout the book, and in many cases the notation is so complicated that readers frequently lose interest before the most important parts are reached.

In most of the subsequent contributions the idea of point or plane coördinates is not considered; most of them presuppose a knowledge of the relation between these and the quantities which directly define a straight line, while others define the new coördinates as parameters, without discussing their meaning.

In view of the great interest which the subject has awakened and its fruitful application to the study of curves and surfaces, its analogy to the geometry of the sphere and its assistance to mechanics, one wonders why no elementary and systematic treatise on the subject has appeared. The work of Sturm is the only comprehensive work on line geometry that has as yet appeared, and it is by no means an elementary one. It is purely synthetic in its treatment

and presupposes much that is not generally known. The German translation of Salmon, by Professor Fiedler, gives a short introduction, and a more extensive one is found in the second volume of Clebsch-Lindemann.

In 1887 Professor Koenigs commenced a serial paper in the Annales de Toulouse and the book under consideration is the part of that series which appeared in tomes 3, 6 and 7.

The work begins with a very brief historical introduction and the statement that three phases of line geometry are marked by the works of Plücker, Klein and Lie. Since this chapter was written the work of Sturm has appeared, which distinctly represents a new phase. Chapter I. defines the coördinates of a line as the six determinants of the second order formed by the homogeneous coördinates of two points

$$x_1, x_2, x_3, x_4 ; \; y_1, y_2, y_3, y_4,$$

viz:

$$p_{ik} = x_i y_k - x_k y_i \qquad i, k = 1, 2, 3, 4, i \neq k;$$

or of two planes

$$u_1, u_2, u_3, u_4 ; \; v_1, v_2, v_3, v_4 ;$$

$$q_{ik} = u_i v_k - u_k v_i.$$

$$P \equiv p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} \equiv Q \equiv q_{12}q_{34} + q_{13}q_{42} + q_{14}q_{23} \equiv 0.$$

$$p_{ik} = \frac{\partial Q}{\partial q_{ik}} ; \; q_{ik} = \frac{\partial P}{\partial p_{ik}}.$$

The $p_{ik}, q_{em}$ are denoted by $r_{ik}$, and the quadratic identity by $\omega(r) = 0$; from now on, point and plane coördinates are entirely subordinated.

The analysis of linear transformation of the $r_{ik}$ is next treated, so that all the subsequent discussions are independent of the form of $\omega(r)$; the representation of pencils and hyper-pencils is next obtained, the latter term denoting all the lines in a plane, or all the lines through a point. The chapter closes with the definitions of complex, congruence and series and the geometrical criteria for their degree and class.

Chapter II. begins with an admirably written discussion of poles and polar planes of linear complexes and of conjugate lines; the invariant of the complex is derived and the conjugate of a given line found, with regard to a given linear complex.

Chapter III. is called systems of linear complexes. The first treatment, which is purely analytic, is of the point and plane homography on a straight line. The idea of involution is first introduced in an unhappy way, as the author develops into the principal thought an analogy which Klein had found to exist between linear complexes in involution and metrical geometry of the point.

The usual idea of involution is introduced later, without showing that the two are really identical.

There is a strange intermixture of analytic and geometric methods throughout the chapter; that a congruence has in general two directrices is proven geometrically, while the particular forms are treated algebraically only.

Two features are worthy of remark; the consistent use of partial differential equations, and the discussion of a system of complexes in involution. At the end of the chapter are added, in tangential coördinates, the equations of a special complex, special congruence, special series and four complexes having two coincident lines in common. Chapter IV. is devoted to infinitesimal geometry; it begins with the series, or a system of lines having but one parameter. The osculating hyperbolæ of a ruled surface are treated at length, and the order of contact which a line may have with the surface is clearly put. The condition that the surface is developable is obtained but not discussed. Here may be pointed out an error on page 61, where the author has confused the direct and the complementary systems of linear complexes which are tangent, at a given line, to the surface. Throughout this chapter the plane of presupposed knowledge is much higher than in any other in the book; many theorems are merely cited without proof or reference.

It is shown that any line having two parameters has an envelope, which serves to define the focal surface of a congruence. This surface and the surface of singularities are merely proved to exist; no discussion of either being added. Series, expressed by complexes, are not discussed at all.

The congruence is taken up again, defined by two parameters and some differential expressions are derived, the principal one being the condition which must be fulfilled when a congruence has an osculating linear complex.

The next chapter, V., comprises about two-fifths of the entire book. It is devoted to two problems; linear transformation of the coördinates and the analogy between line geometry and stereographic projection in four dimensions. Only two sets of transformations are considered; those in which the quadratic identity retains the form

$$x_1x_4 + x_2x_5 + x_3x_6 \equiv 0,$$

and those which satisfy the form

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 \equiv 0.$$

The first may be called tetrahedral coördinates; the axes of the six special complexes $x_i = 0$ are the edges of a tetrahedron. The second system, which are named Klein's coördinates, is composed of complexes mutually in involution. By a happily chosen nomenclature these complexes, their congruences and their hyperboloids are beautifully discussed; it is to be regretted that the author makes no application of them to the quadratic complex.

The last subject treated is the transformation which changes the lines of a complex into the points of a sphere in space of four dimensions, then this sphere is stereographically projected upon the ordinary sphere. In this connection the author uses geometrical figures; the other chapters are entirely without them.

One gathers, from various remarks made in the course of the work, that the author had intended to make the treatise much more extensive, especially as the second part of the title is entirely ignored.

Roughly, the book is a reproduction, with some extensions, and some omissions, of parts of three papers by Professor Klein; those in *Math. Ann.*, vol. 2, pp. 203–213, vol. 5, pp. 257–268, and pp. 278–293.

Although the book is the only elementary treatise on the subject, two important fundamental subjects have been entirely omitted, viz., the canonical form of the linear complex,

$$x_1 = kx_2,$$

and the algebraic discussion of the quadratic complex, as given by Klein. Should one use the book to enable him to better understand most of the memoirs on line geometry, it would prove a valuable aid, but read alone, the reader would get but a narrow and one-sided idea of its usefulness.

VIRGIL SNYDER.

CORNELL UNIVERSITY,
*August 24*, 1897.

## NOTES.

UNIVERSITY OF BERLIN. During the coming winter semester the following mathematical courses will be offered: By Professor FUCHS: Theory of the partial differential equations; Theory of hyperelliptic functions. By Professor SCHWARZ: Integral calculus; Selected chapters from theory of analytical functions; Theory of elliptic functions; Mathematical colloquium. Professor FROBENIUS: Analytical geometry; Theory of numbers. Professor HELMERT: Method of least squares. Professor HETTNER: Calculus of probabilities and theory of errors. Professor KNOBLAUCH: Differential calculus; Theory of partial differential equations with application to theory of surfaces and mechanics; Exercises in differential calculus. Professor HENSEL; Algebra; Theory of algebraic functions of one variable and Abelian integrals; Analytical introduction to the theory of Abelian integrals. Professor LEHMANN-FILHÉS: Problem of three bodies; Method of least squares. DR. HOPPE: Analytical geometry (by text-book); Differential calculus and series (by text-book).

AMONG B. G. Teubner's recent announcements of forthcoming books the most important is that of the "Encyclopädie der mathematischen Wissenschaften, mit Einschluss der Anwendungen" (see BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, 2d Series, Vol. 3, p. 356), edited, with the aid of the academies of sciences of Munich, Vienna, and Göttingen, by Professors H. BURKHARDT and W. F. MEYER. The object of this work is to give, in a form at once concise and convenient for reference, but with the greatest possible completeness, a comprehensive presentation of the mathematical sciences in their present *results*, and at the same time, by means of careful references, an historical survey of the development of mathematical *methods* since the beginning of the 19th century. The applications to mechanics, physics, astronomy, geodesy, the engineering sciences will be included so that, on the one hand, the mathematician may readily see what questions are put to him by the applied sciences, while, on the other, the astronomer, the physicist, the engineer will be able to find the answer given by mathematics to his questions. The arrangement of the matter will be by subjects in the first five volumes; the sixth and last will provide an exhaustive alphabetical index. The editors will be assisted by a com-

mittee representing the three academies interested and consisting of Professors W. Dyck (Munich), G. von Escherich and L. Boltzmann (Vienna), F. Klein (Göttingen), and H. Weber (Strassburg).

THE same firm has in preparation the second and last volume of M. Krause's "Theorie der doppelt-periodischen Funktionen einer veränderlichen Grösse," and a German translation, by A. Schepp, of E. J. Routh's "Rigid Dynamics," in two volumes, with a preface by Professor Klein; finally two volumes entitled "Urkunden zur Geschichte der nichteuklidischen Geometrie," by Professors Engel and Stäckel. The last work is devoted to Lobachevsky and the Bolyais, father and son; besides biographies of these men, the work will contain translations from the Russian of Lobachevsky's "On the foundations of geometry" and "New elements of geometry with a complete theory of parallels," not hitherto translated, W. Bolyai's "Theoria parallelarum" (1804), J. Bolyai's "Appendix" and parts of the "Tentamen."

LITHOGRAPHIC copies of Professor Felix Klein's lectures at the University of Göttingen have been obtainable for several years from Dr. Fr. Schilling, Karlsruhe, Baden. Since our former note some additions have been made to the set, which has also been revised by the substitution of recent lectures for earlier ones on the same subjects, and now consists of the following six volumes, covering the period 1889–1896:] I. Ausgewählte Kapitel aus der Zahlentheorie, two parts, 1895–6 (Mk. 12.00); II. Lineare Differentialgleichungen der zweiten Ordnung, 1894 (Mk. 7.00); III. Ueber die hypergeometrische Funktion, 1893–4 (Mk. 7.50); IV. Höhere Geometrie, two parts, 1892–3 (Mk. 12.50); V. Riemann'sche Flächen, two parts, 1891–2 (Mk. 10.00); VI. Nicht-Euklidische Geometrie, two parts, 1889–90 (Mk. 11.50). A summary of contents of the several lectures is found in Professor Klein's notices in vols. 45, 46 and 48 of the *Mathematische Annalen*. The lithographed volumes are still furnished by Dr. Schilling at the prices annexed above. In a short time, however, the firm of B. G. Teubner will take charge of their issue, and the prices will be increased.

THE MACMILLAN COMPANY, of New York, have in press a translation, by Professor Holgate, of Reye's "Geometrie der Lage," part I.

MR. WILLIAM ESSON has been appointed Savilian professor of mathematics at Oxford, to succeed the late Professor J. J. Sylvester.

DR. LUDWIG SCHLESINGER, who was recently called from Berlin to an associate professorship at the University of Bonn, has been appointed professor of mathematics at the University of Klausenburg, in Transylvania, to succeed the late Professor Ludwig Martin.

DR. ARTEMAS MARTIN, of Washington, D. C., has been recently promoted to be chief of the Library and Archives Division in the office of the U. S. Coast and Geodetic Survey.

MR. P. A. LAMBERT has been promoted to an assistant professorship of mathematics at Lehigh University.

AMONG recent academic appointments are the following: —LEONARD E. DICKSON, Ph.D. (Chicago), to an instructorship in mathematics at the University of California; CHAS. EDWARD ST. JOHN, Ph.D. (Harvard), to an associate professorship of physics and mathematics at Oberlin College; MILTON BROCKETT PORTER, Ph.D. (Harvard), to an instructorship of mathematics at the University of Texas; EDWARD V. HUNTINGDON, Ph.D. (Harvard), to an instructorship in mathematics at Williams College.

MR. JOSEPH COTTIER, fellow in mechanics at Columbia University, died in Paris, France, August 17th, of typhoid fever.

# NEW PUBLICATIONS.

## I. HIGHER MATHEMATICS.

AMBERG (E. J.). Ueber den Körper dessen Zahlen sich rational aus zwei Quadratwurzeln zusammensetzen. [Diss.] Zürich, 1897. 8vo. 50 pp.

BAER (K.). Ueber das logarithmische Potential einer Pascal'schen Schnecke. [Progr.] Kiel, 1897. 4to. 24 pp., 1 plate.

BERTEZENE (A.). Mémoire à l'Académie des sciences: (1) Des probabilités; (2) théorèmes sur le jeu; (3) théorie mathématique du baccarat; (4) sommation de nombres; (5) problèmes sur les pôles et polaires. Suivi de morceaux choisis du poème "La révolution," et d'une page d'histoire "Chute de M. Thiers." Paris, Voix de Paris, 1897. 18mo. 87 pp.

BIANCHI (L.). Teoria dei gruppi e delle equazioni algebriche secondo Galois. Lezioni fatte nella R. Scuola normale superiore di Pisa 1896–97 raccolte e pubblicate per cura di V. Boccara. Pisa, 1897. 8vo. 376 pp. Fr. 13.00

BOCCARA (V.). See BIANCHI (L.).

BOLYAI DE BOLYA (J.). Appendix, Scientiam Spatii absolute veram exhibens, a veritate aut falsitate axiomatis XI Euclidei (a priori haud unquam decidenda) independentem; adjecta ad casum falsitatis quadratura circuli geometrica. Cum traductione Hungarica et biographia ediderunt J. Suták et F. Schmidt. Budapestini, 1897. 8vo. 28 and 143 pp., 2 plates. Mk. 5.00

BOULANGER (A.). Contribution à l'étude des équations différentielles linéaires et homogènes intégrables algébriquement. [Thèse.] Paris, Gauthier-Villars, 1897. 4to. 129 pp.

BRÜCKNER (J. M.). Geschichtliche Bemerkungen zur Aufzählung der Vielflache. [Progr.]. Zwickau, 1897. 4to. 19 pp., 7 plates.

CAUCHY (A.). Oeuvres complètes, publiées sous la direction scientifique de l'Académie des sciences et sous les auspices de M. le ministre de l'instruction publique. 2me série. Vol. III: Cours d'analyse de l'École Royale Polytechnique. Partie 1: Analyse algébrique. (1821.) Paris, Gauthier-Villars, 1897. 4to. 476 pp. Fr. 25.00

DISCHNER (H.). Ein Beispiel für den Zusammenhang des allgemeinen Integrals einer linearen homogenen Differentialgleichung mit einem Fundamentalsystem von Integralen. [Progr.] Erfurt, 1897. 4to. 15 pp.

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JAHRBUCH über die Fortschritte der Mathematik, begründet von C. Ohrtmann. Herausgegeben von E. Lampe. Vol. 26: Jahrgang 1895. (In 3 Heften.) Heft 1. Berlin, Reimer, 1897. 8vo. 528 pp. Mk. 13.50

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MILLER (G. A.). The non-regular transitive substitution groups whose order is the product of three unequal prime numbers. (*Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, Jahrgang 42, 1897.) 8vo. pp. 68–73.

MOHRMANN. Bestimmung der Koeffizienten in der Potenzreihe, die durch Umkehrung einer angegebenen Potenzreihe entsteht, mit Anwendungen. I. [Progr.] Arnstadt, 1897. 4to. 36pp. Mk. 1.50

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SOMMERFELD (A.). See KLEIN (F.).

SIMONIN (M.). Sur l'orbite de 108 Hécube. [Thèse.] Paris, Gauthier Villars, 1897. 4to. 79 pp.

SINRAM (A.). Fragmente zum kosmischen Bewegungsgesetz (Incitations-Theorie) und zur Mechanik des Himmels. Hamburg, Gräfe and Sillem, 1897. 8vo. 32 pp. Mk. 1.00

STEINMETZ (C. P.) and BERG (E. J.). Theory and calculation of alternating current phenomena. New York and London, 1897. 8vo. 18 and 431 pp. Cloth. $3.00

SWYNGEDAUW (R.). Contribution à l'étude des décharges. Les potentiels explosifs statique et dynamique. [Thèse.] Paris, Carré, 1897. 4to. 41 pp. Fr. 4.50

THIEM (W.). Ueber die Magnetisierungsarbeit pulsierender Gleichströme im Vergleich mit aequivalentem Wechselstrome. [Diss.] Rostock, 1896. 8vo. 39 pp., 5 plates.

VALLEREY (J.). Géométrie et mécanique pour les capitaines de commerce et les candidats aux examens de la marine marchande. Paris, 1897. 8vo. Cloth. Fr. 8.50

WALTER (A.). Invarianten und elliptische Modulfunktionen auf thermochemischem Gebiete. Ein Beitrag zur Lehre von der chemischen Energie. [Progr.] Tarnowitz, 1897. 4to. 40 pp.

WATELET (E.). Dessin géométrique, 2 parties. Paris, 1897. 4to. Fr. 10.50

WEILER (W.). Die Dynamomaschine. Physikalische Principien, Arten, Teile, Wechselwirkung der Teile und Construction derselben. 3te Auflage. Magdeburg, 1897. 8vo. 16 and 199 pp. Mk. 4.00

WILCZYNSKI (E. J.). Hydrodynamische Untersuchungen mit Anwendungen auf die Theorie der Sonnenrotation. Berlin, 1897. 4to. 34 pp. Mk. 1.50

ZEHNDER (L.). Die Mechanik des Weltalls in ihren Grundzügen dargestellt. Freiburg, 1897. 8vo. 7 and 176 pp. Mk. 3.00

ZÜRN (H.). Das Pendel und seine Verwendung. Plauen, 1897. 4to. 39 pp. Mk. 1.50